Quantum work-fluctuation theorem: Exact case study for a perturbed oscillator

Takaaki Monnai*

Department of Applied Physics, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

(Received 11 June 2009; revised manuscript received 22 October 2009; published 22 January 2010)

We analytically explore the fluctuation of the work-induced entropy production of an externally perturbed quantum harmonic oscillator interacting with several reservoirs. The quantum fluctuation of the work amounts to a non-Gaussian fluctuation of the entropy production, which interpolates the Gaussian and the Poissonian distributions at high- and low-temperature regimes. Also, it is shown that the corresponding fluctuation theorem symmetry is rigorously satisfied.

DOI: 10.1103/PhysRevE.81.011129 PACS number(s): 05.40.−a, 05.30.−d

I. INTRODUCTION

In mesoscopic systems, dynamical variables show significant fluctuations around its averages. Actually, a universal symmetry called fluctuation theorem has been established for various classical dynamics both for deterministic systems [1–5] and locally detailed balanced stochastic systems [6–15]. With suitable definitions of entropy production, fluctuation theorem has been confirmed also in nonequilibrium reaction theories [16–18]. The connection between fluctuation theorem and the nonlinear response theory was also intensively surveyed [18].

On the other hand, the quantum extension of the fluctuation theorem has been received considerable attentions [19–32]. The main subject has been the derivation of the symmetry relation by a suitable treatment of the quantum fluctuation or the measurement schemes of the entropy production. Among the various schemes for evaluating the entropy production [19,22,30,32], two-point measurement approach composed of the initial determination of the state, the unitary time evolution, and the measurement of energy or particle-number at the final time is well established. Two-point measurement scheme reproduces the classical symmetry of fluctuation theorem and full counting statistics in the semiclassical regime [33]. Also, it is rigorously shown that the characteristic function for the two-point measurement depends only on the thermodynamic forces due to the conservation law for a long time [22]. For further insights of two-point measurement scheme, it is advantageous to more explicitly evaluate the characteristic function.

The purpose of this paper is to envisage whether the work-induced entropy production due to the external perturbation satisfies the fluctuation theorem for the two-point measurement scheme, and to investigate how the quantum fluctuation affects the distribution of the corresponding entropy production compared to the classical case. In order to make clear the role of the external perturbation, we explore a model, which is driven out of equilibrium also by thermodynamic forces. For an exactly solvable model of a damped harmonic oscillator interacting with several large reservoirs, we concern with the work-induced entropy production, and calculate its probability distribution. Especially, it is shown that the quantum fluctuation amounts to a non-Gaussian probability distribution, which interpolates the Gaussian- and Poissonian-behaviors at high- and low-temperature regimes. Fluctuation theorem is satisfied for this probability distribution.

This paper is organized as follows. Section II describes our model of damped harmonic oscillator, and the time-dependent total Hamiltonian is diagonalized. Also, the initial stationary state is derived from a local equilibrium ensemble at a remote past time. In Sec. III, we define the work-induced entropy production due to the external perturbation and present its characteristic function. In Sec. IV, we analytically calculate the characteristic function of the work-induced entropy production, and explore physically the statistical properties of the consequent non-Gaussian distribution.

II. MODEL

We analytically investigate the entropy production at a nonequilibrium state of a temporally perturbed harmonic oscillator interacting with N large reservoirs $r_1, \dots, r_N$. Note that hereafter, the total system including the reservoirs is considered as isolated except for the external perturbation. As ideal reservoirs, we consider assemblies of harmonic oscillators [35–38]. For the case of a single reservoir, our model can be interpreted as a quantum counterpart of the experiment on a colloidal particle dragged in water [34]. Our scheme will be useful for small object confined in a harmonic potential interacting with several electromagnetic fields at different temperatures [38]. In the isothermal case, there is no thermodynamic force such as temperature gradient, and it has been shown that work-induced entropy production satisfies fluctuation theorem [11–13].

Let us consider the total Hamiltonian

$$\hat{H}(t) = \hbar\Omega\hat{a}^{+}\hat{a} - \sqrt{\hbar}(\hat{a}+\hat{a}^{+})f(t) + \sum_{j=1}^{N}\int dk\hbar\omega_{kj}\hat{a}_{kj}^{+}\hat{a}_{kj} + \hbar\sum_{j=1}^{N}\int dk(u_{kj}\hat{a}\hat{a}_{kj}^{+} + v_{kj}\hat{a}_{kj}\hat{a}^{+}). \quad (1)$$

Here, Ω and ω_{kj} are the natural frequencies of the system and jth reservoir oscillators with the annihilation operators $\hat{a}$ and $\hat{a}_{kj}$, which satisfies $[\hat{a},\hat{a}^{+}]=1$, $[\hat{a},\hat{a}_{kj}^{+}]=0$, and $[\hat{a}_{ki},\hat{a}_{k'j}^{+}] = \delta_{ij}\delta(k-k')$, respectively. The interaction between the sys-

*monnai@aoni.waseda.jp

tem and each jth reservoir is bilinear where the coupling strengths satisfy $u_{kj}=v_{kj}^*$. $f(t)$ presents the position of the potential, which is initially located at the origin $f(0)=0$. The external perturbation $\sqrt{\hbar}(\hat{a}+\hat{a}^+)f(t)$ is interpreted as the potential dragging with the velocity $\dot{f}(t)$. Indeed, $a+a^+$ corresponds to the position coordinate, and the energy stored in the harmonic potential located at $f(t)$ is proportional to $[a+a^+-\sqrt{\hbar}f(t)]^2$, and after the rotating wave approximation, the operator part gives the perturbation term [35,38]. At the initial time $t=0$, the center of the harmonic potential f starts to move.

A. Normal modes

In this section, we derive the normal modes, which completely characterize the time evolution in the presence of the external perturbation. Here, we show the outline of the derivation, and for more details see also the similar analysis [36] for the unperturbed case. First, we shall diagonalize the total Hamiltonian by the normal modes $\hat{\alpha}_{kj}$ in the absence of external perturbation $f(t)\equiv 0$, i.e., $\hat{H}=\hat{H}(0)$. Since the interaction term $\sum_{j=1}^N \int dk\hbar(u_{kj}aa_{kj}^+ + v_{kj}a_{kj}a^+)$ is quadratic, the creation of more than two particles from a particle is not allowed. Therefore, the frequency renormalization is absent. This observation is justified by the calculation of the normal modes described below. The total Hamiltonian is diagonalized as $\hat{H}=\sum_{j=1}^N \int dk\hbar\omega_{kj}\hat{\alpha}_{kj}^+\hat{\alpha}_{kj}$.

Next, we determine the normal modes. Since the equations of motion are linear

$$\dot{\hat{a}} = -i\Omega\hat{a} - i\sum_{j=1}^N \int dk v_{kj}\hat{a}_{kj},$$

$$\dot{\hat{a}}_{kj} = -i\omega_{kj}\hat{a}_{kj} - iu_{kj}\hat{a}, \tag{2}$$

the normal modes $\hat{\alpha}_{kj}$ are given as a linear combination of these variables

$$\hat{\alpha}_{kj} = \hat{a}_{kj} + b_{kj}\hat{a} + \sum_{m=1}^N \int dk' c_{kk'jm}\hat{a}_{k'm}. \tag{3}$$

Due to the linearity, the spectral distribution of the frequencies ω_{kj} is the same for $\hat{a}_{kj}$ and $\hat{\alpha}_{kj}$.

Let us determine the coefficients b_{kj} and $c_{kk'jm}$. The time derivative of $\hat{\alpha}_{kj}$ is calculated as

$$\dot{\hat{\alpha}}_{kj} = \frac{1}{i\hbar}\left[\hat{\alpha}_{kj}, \sum_{j=1}^N \int dk'\hbar\omega_{k'j}\hat{\alpha}_{k'j}^+\hat{\alpha}_{k'j}\right] = -i\omega_{kj}\hat{\alpha}_{kj}. \tag{4}$$

On the other hand, the time derivative of Eq. (3) is

$$\dot{\hat{\alpha}}_{kj} = \dot{\hat{a}}_{kj} + b_{kj}\dot{\hat{a}} + \sum_{m=1}^N \int dk' c_{kk'jm}\dot{\hat{a}}_{k'm}. \tag{5}$$

Equating Eqs. (4) and (5) and substituting Eq. (2), the coefficients are determined as

$$b_{kj} = -\frac{u_{kj}}{\eta_-(\omega_{kj})},$$

$$c_{kk'jm} = -\frac{u_{kj}v_{k'm}}{\eta_-(\omega_{kj})(\omega_{kj}-\omega_{k'm}-i0)}, \tag{6}$$

where the dispersion function $\eta_\pm(\omega)$ is defined as

$$\eta_\pm(\omega) = \Omega - \omega + \sum_{n=1}^N \int dk' \frac{|u_{k'n}|^2}{\omega-\omega_{k'n}\pm i0}. \tag{7}$$

The sign of infinitesimally small imaginary part $\pm i0$ is chosen so that the solution of the outgoing wave is selected. Note that the dispersion function $\eta_\pm(\omega)$ has no zeros, and there are no bound states. The normal modes are thus given as

$$\hat{\alpha}_{kj} = \hat{a}_{kj} - \frac{u_{kj}}{\eta_-(\omega_{kj})}\hat{a} - \sum_{m=1}^N u_{kj}\int dk' \frac{v_{k'm}}{\omega_{kj}-\omega_{k'm}-i0}\frac{\hat{a}_{k'm}}{\eta_-(\omega_{k'j})}. \tag{8}$$

In terms of the normal modes, the annihilation operator of the system $\hat{a}$ and those of reservoirs $\hat{a}_{kj}$ are also expressed linearly as

$$\hat{a} = \sum_{j=1}^N \int dk \frac{-v_{kj}}{\eta_+(\omega_{kj})}\hat{\alpha}_{kj},$$

$$\hat{a}_{kj} = \alpha_{kj} + \sum_{m=1}^N \int dk' \frac{u_{kj}}{\eta_+(\omega_{k'j})}\frac{v_{k'm}}{\omega_{kj}-\omega_{k'm}}\hat{\alpha}_{k'm}. \tag{9}$$

Let us next survey the externally perturbed case. The equation of motion for $\hat{\alpha}_{kj}(t)$ is

$$\dot{\hat{\alpha}}_{kj}(t) = -i\omega_{kj}\hat{\alpha}_{kj}(t) - \frac{iu_{kj}}{\sqrt{\hbar}\,\eta_-(\omega_{kj})}f(t). \tag{10}$$

The solution is

$$\begin{aligned}\hat{\alpha}_{kj}(t) &= e^{-i\omega_{kj}t}\hat{\alpha}_{kj} - \int_0^t ds \frac{iu_{kj}}{\sqrt{\hbar}\,\eta_-(\omega_{kj})}f(s)e^{-i\omega_{kj}(t-s)} \\ &= e^{-i\omega_{kj}t}\hat{\alpha}_{kj} - \frac{u_{kj}f(t)}{\sqrt{\hbar}\,\omega_{kj}\eta_-(\omega_{kj})} + \int_0^t ds \frac{u_{kj}\dot{f}(s)e^{-i\omega_{kj}(t-s)}}{\sqrt{\hbar}\,\omega_{kj}\eta_-(\omega_{kj})}.\end{aligned} \tag{11}$$

Here, we have abbreviated the normal modes at initial time $t=0$ as $\hat{\alpha}_{kj}(0)=\hat{\alpha}_{kj}$. Integrating by parts the second term of the first equality, and using $f(0)=0$ the second equality is obtained. Then the total Hamiltonian in the presence of the perturbation is diagonalized as

$$\hat{H}(t) = \sum_{j=1}^N \int dk\hbar\omega_{kj}\hat{A}_{kj}^+(t)\hat{A}_{kj}(t) - \sum_{j=1}^N \int dk \frac{|u_{kj}|^2}{\omega_{kj}|\eta_+(\omega_{kj})|^2}f(t)^2, \tag{12}$$

with the time-dependent normal modes

$$\hat{A}_{kj}(t) = e^{-i\omega_{kj}t}\hat{\alpha}_{kj} + \int_0^t ds \frac{u_{kj}}{\sqrt{\hbar}\,\eta_-(\omega_{kj})\omega_{kj}} e^{-i\omega_{kj}(t-s)}\dot{f}(s). \tag{13}$$

Note that the last term of Eq. (12) does not appear in the analysis of the entropy production given below. As explained just below Eq. (1), the external perturbation $\sqrt{\hbar}(a+a^{+})$ derives from the energy stored in the harmonic potential as the operator part, and the term proportional to $f(t)^2$ expresses the c-number part which should be removed from Eq. (12) [35]. This term does not appear in the position representation [19].

B. Initial state

Here, we introduce the initial state at $t=0$. For this purpose, we start with a local equilibrium state at far past time $t=-T$ and formally construct a stationary state at $t=0$. For more details of underlying mathematics, see also an algebraic approach [25]. Suppose at a far past time $t=-T$, the system and the reservoirs are prepared as mutually different equilibrium states

$$\hat{\rho}(-T) = \hat{\rho}_s(-T) \otimes \prod_{j=1}^{N} \hat{\rho}_j(-T), \tag{14}$$

with

$$\hat{\rho}_s(-T) = \frac{e^{-\beta\hbar\Omega\hat{a}^{+}\hat{a}}}{Z},$$

$$\hat{\rho}_j(-T) = \frac{1}{Z_j}\exp\left(-\beta_j\int dk\hbar\omega_{kj}\hat{a}_{kj}^{+}\hat{a}_{kj}\right), \tag{15}$$

where β and β_j are the inverse temperatures of the system and the jth reservoir, and Z and Z_j present the partition functions of equilibrium.

Then, let us evolve $\hat{\rho}(-T)$ under the unperturbed dynamics $f(t)\equiv 0$ until $t=0$. In the course of time evolution, a nonequilibrium stationary state would be reached. The stationary state is considered to be characterized only by inverse temperatures of reservoirs β_j, and that of the system β would not appear. In order to clarify this point, we calculate the stationary distribution with the use of normal modes (9). Equation (15) is rewritten as

$$\hat{\rho}_s(-T) = \frac{1}{Z}\exp\Bigg(-\beta\hbar\Omega\sum_j\int dk \times\sum_{j'}\int dk'\frac{v_{kj}u_{k'j'}}{\eta_+(\omega_{kj})\,\eta_-(\omega_{k'j'})}\hat{\alpha}_{kj}^{+}\hat{\alpha}_{k'j'}\Bigg),$$

$$\hat{\rho}_j(-T) = \frac{1}{Z_j}\exp\Bigg[-\int dk\beta_j\hbar\omega_{kj}\Bigg(\hat{\alpha}_{kj}^{+} + \sum_{m=1}^{N}\int dk' \times\frac{v_{kj}}{\eta_+(\omega_{k'j})}\frac{u_{k'm}}{\omega_{kj}-\omega_{k'm}}\hat{\alpha}_{k'm}^{+}\Bigg) \times\Bigg(\hat{\alpha}_{kj} + \sum_{m=1}^{N}\int dk'\frac{u_{kj}}{\eta_-(\omega_{k'j})}\frac{v_{k'm}}{\omega_{kj}-\omega_{k'm}}\hat{\alpha}_{k'm}\Bigg)\Bigg]. \tag{16}$$

For the unperturbed dynamics, the normal mode at $t=-T$ evolves as $U_T\hat{\alpha}_{kj}U_T^{+}=e^{i\omega_{kj}T}\hat{\alpha}_{kj}$, where U_t is the unitary evolution operator of the state during time t. With the use of this relation, we calculate a characteristic function, i.e., the mean of $e^{i(c^{*}a(0)+ca(0)^{+})}$ with an arbitrary number c, to determine the system state at $t=0$

$$\begin{aligned}
&\mathrm{Tr}\,\hat{\rho}_s(-T)e^{i[c^{*}a(0)+ca(0)^{+}]}\\
&\quad=\mathrm{Tr}\,\hat{\rho}_s(-T)U_T^{+}e^{i[c^{*}a(-T)+ca(-T)^{+}]}U_T\\
&\quad=\mathrm{Tr}\,\hat{\rho}_s(-T)\exp\Bigg\{i\Bigg[c^{*}\sum_{j=1}^{N}\int dk\frac{-v_{kj}}{\eta_+(\omega_{kj})}\\
&\qquad\times e^{-i\omega_{kj}T}\alpha_{kj}+(\mathrm{H.c.})\Bigg]\Bigg\}\to 1(T\to\infty).
\end{aligned} \tag{17}$$

Here, we used that as T goes to infinity, the exponential terms $e^{\pm i\omega_{kj}T}$ becomes a rapidly oscillating function of frequency. This fact suggests that the system density matrix is described by unity at the stationary state $t=0$. The density matrix of the reservoirs evolve as

$$\begin{aligned}
U_T\rho_j(-T)U_T^{+}=\frac{1}{Z_j}\exp\Bigg[-\beta_j\hbar\omega_{kj}\int dk\Bigg(&e^{-i\omega_{kj}T}\hat{\alpha}_{kj}^{+}\\
&+\sum_{m=1}^{N}\int dk'\frac{v_{kj}}{\eta_+(\omega_{k'j})}\frac{u_{k'm}}{\omega_{kj}-\omega_{k'm}}e^{-i\omega_{k'm}T}\hat{\alpha}_{k'm}^{+}\Bigg)\\
&\times\Bigg(e^{i\omega_{kj}T}\hat{\alpha}_{kj}+\sum_{m=1}^{N}\int dk'\\
&\times\frac{u_{kj}}{\eta_-(\omega_{k'j})}\frac{v_{k'm}}{\omega_{kj}-\omega_{k'm}}e^{i\omega_{k'm}T}\hat{\alpha}_{k'm}\Bigg)\Bigg].
\end{aligned} \tag{18}$$

In the exponent of $U_T\rho_j(-T)U_T^{+}$, only the diagonal term remains, and we have $U_T\rho_j(-T)U_T^{+}\to\frac{1}{Z_j}\times\exp(-\beta_j\int dk\hbar\omega_{kj}\hat{\alpha}_{kj}^{+}\hat{\alpha}_{kj})$ as $T\to\infty$.

Therefore, we choose the stationary state at $t=0$ as the initial state. The density matrix of *the total system* is prepared as the product of canonical distributions of the normal modes $\hat{A}_{kj}(0)=\hat{\alpha}_{kj}$ at mutually different inverse temperatures β_j, and partition functions Z_j,

$$\hat{\rho}(0) = \frac{1}{Z_1 Z_2 \cdots Z_N} \exp\left[-\sum_{j=1}^{N} \beta_j \int dk\hbar\omega_{kj} \hat{A}^{+}_{kj}(0)\hat{A}_{kj}(0) \right]. \tag{19}$$

The partition functions Z_j are the same as those of reservoirs at $t=-T$. As already mentioned, the inverse temperature of the system β disappears.

III. FLUCTUATION OF THE WORK-INDUCED ENTROPY PRODUCTION

In order to define the entropy production, we decompose the total Hamiltonian as

$$\sum_{j=1}^{N} \int dk\hbar\omega_{kj} \hat{A}^{+}_{kj}(t)\hat{A}_{kj}(t) = \sum_{j=1}^{N} \hat{H}_j(t),$$

$$\hat{H}_j(t) \equiv \int dk\hbar\omega_{kj} \hat{A}^{+}_{kj}(t)\hat{A}_{kj}(t). \tag{20}$$

The work externally done on the system is eventually transferred to $\hat{H}_j$. Note that $\hat{H}_j$ is considered as Hamiltonian of a subsystem including the reservoir r_j, and Eq. (20) is a decomposition to mutually independent subsystems $[\hat{H}_i, \hat{H}_j] = 0$. Let us denote the macroscopic eigenstates specified by eigenenergies as $\hat{H}_j(s)|E_j(s)\rangle = E_j(s)|E_j(s)\rangle$. It is remarked that $|E_j(s)\rangle$ is composed of many Fock states, but in realistic situation the determination of energy would be possible within experimental precision. Therefore, we here assume that projective measurement of eigenenergy is possible. In this sense, the eigenenergy $E_j(s)$ specifies the eigenstate. Throughout this paper, the total system including the reservoirs is considered as an isolated system. Then the eigenenergy change divided by the temperature gives a reasonable estimate of the entropy production. The work-induced entropy production is evaluated as

$$\Sigma_0^t = \sum_{j=1}^{N} \beta_j [E_j(t) - E_j(0)]. \tag{21}$$

Note that the c number of Eq. (12) does not contribute to Σ_0^t. The fluctuation of the entropy production Σ_0^t is defined as [22–25,29,33]

$$\begin{aligned} P(\Sigma_0^t = A) &= \int \frac{d\xi}{2\pi} e^{-i\xi A} \,\mathrm{Tr}\, \hat{\rho}(0) \exp\left[i\xi \sum_{j=1}^{N} \beta_j \hat{H}_j(t) \right] \\ &\quad \times \exp\left[-i\xi \sum_{j=1}^{N} \beta_j \hat{H}_j(0) \right] \\ &= \sum_{\{E_j(0)\},\{E_j(t)\}} \frac{1}{Z_1 Z_2 \cdots Z_N} \\ &\quad \times \exp\left[-\sum_{j=1}^{N} \beta_j E_j(0) \right] |\langle \{E_j(t)\} | \{E_j(0)\} \rangle|^2 \delta(\Sigma_0^t - A), \end{aligned} \tag{22}$$

where $\{E_j(s)\}$ denotes a set of eigenvalues of $\hat{H}_1(s), \ldots, \hat{H}_N(s)$, and $|\{E_j(s)\}\rangle \equiv \Pi_{j=1}^{N} |E_j(s)\rangle$. Also, $\hat{\rho}(0) = \frac{1}{Z_1, \ldots, Z_N} \exp[-\Sigma_{j=1}^{N} \beta_j \hat{H}_j(0)]$ represents the initial ensemble, the transfer amplitude $|\langle \{E_j(t)\} | \{E_i(0)\} \rangle|^2$ accounts for the unitary time evolution, and Dirac delta samples the trajectories with a particular value of entropy production $\Sigma_0^t = A$. Then, the characteristic function is given by the Fourier transform

$$\begin{aligned} \Phi(\xi) &\equiv \int dA e^{i\xi A} P(\Sigma_0^t = A) \\ &= \mathrm{Tr} \frac{1}{Z_1 Z_2 \cdots Z_N} \exp\left[-\sum_{j=1}^{N} \beta_j \hat{H}_j(0) \right] \exp\left[i\xi \sum_{j=1}^{N} \beta_j \hat{H}_j(t) \right] \\ &\quad \times \exp\left[-i\xi \sum_{j=1}^{N} \beta_j \hat{H}_j(0) \right]. \end{aligned} \tag{23}$$

As already shown by several authors [20–24] for the case of single reservoir, the fluctuation theorem holds for the energy change. Here, let us briefly summarize it for the case of multiple reservoirs. First, the characteristic function is evaluated as

$$\begin{aligned} \Phi(\xi) &= \mathrm{Tr} \frac{1}{Z_1 Z_2 \cdots Z_N} \exp\left[-\sum_{j=1}^{N} \beta_j \hat{H}_j(0) \right] \\ &\quad \times \exp\left[i\xi \sum_{j=1}^{N} \beta_j \hat{H}_j(t) \right] \exp\left[-i\xi \sum_{j=1}^{N} \beta_j \hat{H}_j(0) \right] \\ &= \mathrm{Tr} \frac{1}{Z_1 Z_2 \cdots Z_N} \exp\left[i\xi \sum_{j=1}^{N} \beta_j \hat{H}_j(t) \right] \\ &\quad \times \exp\left[-\sum_{j=1}^{N} \beta_j (1 + i\xi) \hat{H}_j(0) \right]. \end{aligned}$$

Then, by the change in the variable $\xi' = \xi - i$,

$$\begin{aligned} \Phi(\xi) &= \mathrm{Tr} \frac{1}{Z_1 Z_2 \cdots Z_N} \exp\left[-\sum_{j=1}^{N} \beta_j \hat{H}_j(t) \right] \exp\left[\vphantom{\sum_{j=1}^{N}} \right. \\ &\quad \left. - i\xi' \sum_{j=1}^{N} \beta_j \hat{H}_j(0) \right] \exp\left[\sum_{j=1}^{N} i\xi' \beta_j \hat{H}_j(t) \right] \equiv \Phi^r(-\xi'), \end{aligned} \tag{24}$$

where Φ^r is the characteristic function for the energy change in the time-reversed dynamics evolving with reversed perturbation defined as $f^r(s) \equiv f(t-s)$, and starts from the ensemble $\frac{1}{Z_1 \cdots Z_N} \exp[-\Sigma_{j=1}^{N} \beta_j \hat{H}_j(t)]$. Then, the fluctuation of the energy change divided by the temperature Σ_0^t obeys the symmetry

$$\begin{aligned} P(\Sigma_0^t = A) &= \int \frac{d\xi}{2\pi} e^{-i\xi A} \Phi(\xi) \\ &= \int \frac{d\xi}{2\pi} e^{-i\xi A} \Phi^r(-\xi + i) \\ &= \int \frac{d\xi'}{2\pi} e^{A} e^{-i\xi' A} \Phi^r(-\xi') \end{aligned}$$

$$= e^{A}P(\tilde{\Sigma}_0^t = -A), \quad (25)$$

where $\tilde{\Sigma}_0^t \equiv \sum_{j=1}^N \beta_j[E_j(0)-E_j(t)]$ is the entropy production associated with the time-reversed dynamics.

IV. NON-GAUSSIAN CHARACTERISTIC FUNCTION

In our soluble case, the characteristic function is directly calculated. With the use of formula (A2) for a boson operator $\hat{\alpha}$,

$$\mathrm{Tr}\, e^{-\beta\hbar\omega\hat{\alpha}^+\hat{\alpha}} e^{i\xi\beta\hbar\omega(\hat{\alpha}^+ + u)(\hat{\alpha}+v)} e^{-i\xi\beta\hbar\omega\hat{\alpha}^+\hat{\alpha}} = \frac{1}{1-e^{-\beta\hbar\omega}} e^{-uv(e^{i\xi\beta\hbar\omega}-1-[|1-e^{-i\xi\beta\hbar\omega}|^2/(e^{\beta\hbar\omega}-1)])}, \quad (26)$$

the characteristic function turns out to be

$$\log \Phi(\xi) = \sum_{j=1}^N \int_0^t ds \int_0^t ds' \int dk \frac{|u_{kj}|^2}{\hbar\omega_{kj}^2|\eta_+(\omega_{kj})|^2} \dot{f}(s)\dot{f}(s') \times e^{-i\omega_{kj}(s-s')}\left(e^{i\xi\beta_j\hbar\omega_{kj}} - 1 - \frac{|1-e^{-i\xi\beta_j\hbar\omega_{kj}}|^2}{e^{\beta_j\hbar\omega_{kj}}-1}\right). \quad (27)$$

This characteristic function is the main result of the present paper. Interestingly, the characteristic function interpolates the Gaussian and Poissonian distribution at high- and low-temperature regimes.

Before showing this temperature dependence, we have a few remarks on statistical properties.

First, for slowly varying $\dot{f}(t)$ compared to the reservoir variables, $\dot{f}(t)$ would be extracted from the double-integral $\int_0^t ds \int_0^t ds' \frac{|u_{kj}|^2}{\hbar\omega_{kj}^2|\eta_+(\omega_{kj})|^2}\dot{f}(s)\dot{f}(s')e^{-i\omega_{kj}(s-s')}$ in the characteristic function and the integral behaves as $\sin^2(\omega_{kj}/2)t/(\omega_{kj}/2)^2 \sim 2\pi t\delta(\omega_{kj})$ for sufficiently long times. Then the mean entropy production is linear with respect to t.

Second, the characteristic function is invariant under the time reversal operation $\dot{f}\to -\dot{f}$. Then the forward and reversed characteristic function is identical,

$$\Phi(\xi) = \Phi(-\xi + i). \quad (28)$$

Indeed, it is straightforward to verify the symmetry by replacing ξ with $-\xi+i$ in the integrand of Eq. (27)

$$e^{-\beta_j\hbar\omega_{kj}}e^{-i\xi\beta_j\hbar\omega_{kj}} - 1 - \frac{2-(e^{\beta_j\hbar\omega_{kj}}-1)e^{i\xi\beta_j\hbar\omega_{kj}} - (e^{-\beta_j\hbar\omega_{kj}}-1)e^{-i\xi\beta_j\hbar\omega_{kj}} - e^{i\xi\beta_j\hbar\omega_{kj}} - e^{-i\xi\beta_j\hbar\omega_{kj}}}{e^{\beta_j\hbar\omega_{kj}}-1} = e^{i\xi\beta_j\hbar\omega_{kj}} - 1 - \frac{|1-e^{-i\xi\beta_j\hbar\omega_{kj}}|^2}{e^{\beta_j\hbar\omega_{kj}}-1}. \quad (29)$$

From Eq. (28), fluctuation theorem is derived as

$$P(\Sigma_0^t = A) = \int \frac{1}{2\pi} e^{-i\xi A}\Phi(\xi)d\xi = \int \frac{1}{2\pi} e^{-i\xi A}\Phi(-\xi+i)d\xi = e^{A}\int \frac{1}{2\pi} e^{i\xi A}\Phi^r(\xi)d\xi = e^{A}P(\tilde{\Sigma}_0^t = A). \quad (30)$$

From the characteristic function (27), the first and the second cumulants are

$$\log \Phi(\xi) = i\xi m_{0,t} - \frac{\xi^2}{2}\sigma_{0,t}^2 + O(\xi^3),$$

$$m_{0,t} = \sum_{j=1}^N \int_0^t ds \int_0^t ds' \int dk \times \frac{\beta_j|u_{kj}|^2}{\omega_{kj}|\eta_+(\omega_{kj})|^2}\dot{f}(s)\dot{f}(s')\cos \omega_{kj}(s-s'),$$

$$\sigma_{0,t}^2 = \sum_{j=1}^N \int_0^t ds \int_0^t ds' \int dk \times \frac{\beta_j^2\hbar|u_{kj}|^2}{|\eta_+(\omega_{kj})|^2}\coth\frac{\beta_j\hbar\omega_{kj}}{2}\dot{f}(s)\dot{f}(s')\cos \omega_{kj}(s-s'). \quad (31)$$

Similarly, the odd- and even-higher order cumulants are given as

$$\frac{(-1)^{n-1}i\xi^{2n-1}}{(2n-1)!}\sum_{j=1}^N \int_0^t ds \int_0^t ds' \int dk \frac{|u_{kj}|^2}{|\eta_+(\omega_{kj})|^2} \times \beta_j^{2n-1}\hbar^{2n-2}\omega_k^{2n-3}\cos \omega_{kj}(s-s')\dot{f}(s)\dot{f}(s'),$$

$$\frac{(-1)^n\xi^{2n}}{(2n)!}\sum_{j=1}^N \int_0^t ds \int_0^t ds' \int dk \frac{|u_{kj}|^2}{|\eta_+(\omega_{kj})|^2} \times \beta_j^{2n}\hbar^{2n-1}\omega_{kj}^{2n-2}\coth\frac{\beta_j\hbar\omega_{kj}}{2}\cos \omega_{kj}(s-s')\dot{f}(s)\dot{f}(s'). \quad (32)$$

For the high-temperature limit, the mean and the variance become dominant, and the characteristic function tends to a Gaussian distribution satisfying the semiclassical fluctuation dissipation theorem, $2m_{0,t}=\sigma_{0,t}^2+O(\beta^3\hbar^2)$. On the other hand, in the low-temperature regime, the characteristic function behaves as a Poissonian distribution.

For a multicomponent Poissonian distribution of integer variables k_i with arbitrary cumulants $\lambda_i(i=1,2,\ldots)$, let us consider a stochastic variable $\Sigma\equiv\Sigma_i\nu_i k_i$, where the coefficients ν_i are constants which corresponds to $\beta_j\hbar\omega_{kj}$. The variables k_i are interpreted as a number of events which occurs during a time interval. Then the distribution function is

$$\begin{aligned} P(\Sigma=A) &= \sum_{k_1,k_2,\ldots,k_n,\ldots=0}^{\infty} \delta\left(\sum_j \nu_j k_j - A\right)\prod_{i=1}^{\infty}\frac{(\lambda_i)^{k_i}}{k_i!}e^{-\lambda_i} \\ &= \int \frac{d\xi}{2\pi}e^{-i\xi A}\sum_{k_1,k_2,\ldots,k_n,\ldots=0}^{\infty}\prod_{i=1}^{\infty}\frac{(\lambda_i)^{k_i}e^{i\xi\nu_i k_i}}{k_i!}e^{-\lambda_i} \\ &= \int \frac{d\xi}{2\pi}e^{-i\xi A}\exp\left[\sum_n (e^{i\xi\nu_n}-1)\lambda_n\right]. \end{aligned} \tag{33}$$

Here, the second equality is obtained with the use of the formula $\delta(\Sigma_j\nu_jk_j-A)=\int\frac{d\xi}{2\pi}\exp[i\xi(\Sigma_j\nu_jk_j-A)]$. Actually, the last term of Eq. (27) is exponentially small in the low-temperature compared to the other two terms, resulting to this characteristic function given as the integrand. In the present case, the fluctuating quantity Σ just corresponds to the work-induced entropy production Σ_0^t. The Poissonian distribution is intuitively reasonable, since the discreteness of the spectrum for each mode is magnified by the large inverse temperatures, $\beta_j\hbar\omega_{kj}n_{kj}$, where the integer n_{kj} is the eigenvalue of a number operator $a_{kj}^+a_{kj}$.

In summary, we have prepared an initial nonequilibrium stationary state, and externally perturbed the system by a potential dragging. The work done on the system yields thermodynamic entropy production, which satisfies fluctuation theorem. Furthermore, the quantum fluctuation amounts to a non-Gaussian distribution of the entropy production evaluated by the two-point measurement (27), which satisfies quantum fluctuation theorem for the external work-induced entropy production. The distribution (27) interpolates the Gaussian- and Poissonian-behaviors at high- and low-temperature regimes.

V. DISCUSSION

We have explored the fluctuation of the work-induced entropy production in the presence of multiple reservoirs. As a reasonable quantity which characterizes the nonequilibrium state, we defined the energy variations in normal modes divided by the temperatures of the reservoirs, which are identical to that of the entropy production in the stationary state. The energy variations in normal modes $\hat{H}_j(t)$ account for the external work done, and the thermodynamic forces does not appear in the characteristic functions. In this sense, Σ_0^t is work-induced entropy production due to the external driving $\dot{f}$, and the corresponding symmetry is considered as work-fluctuation theorem.

ACKNOWLEDGMENTS

The author is grateful to Professor S. Tasaki, and Dr. D. Andrieux for helpful discussions. This work was partially supported by Waseda University grant for special projects.

APPENDIX: CALCULATION OF THE SINGLE MODE CHARACTERISTIC FUNCTION

In this appendix, we derive formula (26). Let $\hat{\alpha}$ be a boson annihilation operator $[\hat{\alpha},\hat{\alpha}^+]=1$. With the use of coherent states $|\nu\rangle=\Sigma_n\frac{\nu^n}{n!}(\hat{\alpha}^+)^n|0\rangle$, the left-hand side of Eq. (26) is calculated as

$$\begin{aligned} &\sum_n e^{-\beta\hbar\omega\hat{\alpha}^+\hat{\alpha}}e^{u\hat{\alpha}-v\hat{\alpha}^+}e^{i\xi\beta\hbar\omega\hat{\alpha}^+\hat{\alpha}}e^{-u\hat{\alpha}+v\hat{\alpha}^++uv/2}e^{-i\xi\beta\hbar\omega\hat{\alpha}^+\hat{\alpha}} \\ &\quad=\sum_n e^{-(1+i\xi)\beta\hbar\omega n}\langle n|\int d\alpha_1\frac{e^{-|\alpha_1|^2}}{\pi}|\alpha_1\rangle\langle\alpha_1|e^{-u\alpha_1^*+v\alpha_1+(uv/2)}\sum_m|m\rangle\langle m|e^{i\xi\beta\hbar\omega m}\int d\alpha_2\frac{e^{-|\alpha_2|^2}}{\pi}|\alpha_2\rangle\langle\alpha_2|e^{u\alpha_2^*-v\alpha_2}|n\rangle \\ &\quad=\sum_n e^{-(1+i\xi)\beta\hbar\omega n}\int d\alpha_1\int d\alpha_2\frac{e^{-|\alpha_1|^2-|\alpha_2|^2}}{\pi^2}\sum_m\frac{(\alpha_1\alpha_2^*)^n(\alpha_1^*\alpha_2)^m}{n!\,m!}\times e^{-u\alpha_1^*+v\alpha_2-v\alpha_1+u\alpha_2^*+i\xi\beta\hbar m+uv} \\ &\quad=\int\frac{d\alpha_1}{\pi}\int\frac{d\alpha_2}{\pi}e^{-|\alpha_1|^2-|\alpha_2|^2+e^{-(1+i\xi)\beta\hbar\omega}\alpha_1\alpha_2^*+e^{i\xi\beta\hbar\omega}\alpha_1^*\alpha_2-u\alpha_1^*+v\alpha_1+u\alpha_2^*-v\alpha_2}. \end{aligned} \tag{A1}$$

Here, phase variables $\alpha_{1,2}$ run all the points of complex plane. The Gaussian integrals in the last line give

$$\mathrm{Tr}\frac{1}{Z}e^{-\beta\hbar\omega\hat{\alpha}^+\hat{\alpha}}e^{i\xi(\hat{\alpha}^++u)(\hat{\alpha}+v)}e^{-i\xi\beta\hbar\omega\hat{\alpha}^+\hat{\alpha}}=e^{-uv(e^{i\xi\beta\hbar\omega}-1-[|1-e^{-i\xi\beta\hbar\omega}|^2/(e^{\beta\hbar\omega}-1)])}.$$

[1] D. J. Evans, E. G. D. Cohen, and G. P. Morriss, Phys. Rev. Lett. **71**, 2401 (1993).
[2] G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. **74**, 2694 (1995); E. G. D. Cohen and G. Gallavotti, J. Stat. Phys. **96**, 1343 (1999).
[3] G. Gallavotti and E. G. D. Cohen, J. Stat. Phys. **80**, 931 (1995).
[4] D. J. Evans and D. J. Searles, Phys. Rev. E **50**, 1645 (1994); D. Searles and D. J. Evans, J. Chem. Phys. **112**, 9727 (2000); **113**, 3503 (2000); D. J. Evans and D. Searles, Adv. Phys. **51**, 1529 (2002); E. Mittag and D. J. Evans, Phys. Rev. E **67**, 026113 (2003).
[5] C. Jarzynski, J. Stat. Phys. **98**, 77 (2000).
[6] J. Kurchan, J. Phys. A **31**, 3719 (1998).
[7] O. Narayan and A. Dhar, J. Phys. A **37**, 63 (2004).
[8] J. L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).
[9] C. Maes, J. Stat. Phys. **95**, 367 (1999).
[10] T. Taniguchi and E. G. D. Cohen, J. Stat. Phys. **126**, 1 (2007).
[11] S. Terasaki, I. Terasaki, and T. Monnai, e-print arXiv:cond-mat/0208154.
[12] O. Mazonka and C. Jarzynski, Report No. LAUR-99-6303 (unpublished).
[13] R. van Zon and E. G. D. Cohen, Phys. Rev. E **67**, 046102 (2003).
[14] G. E. Crooks, Phys. Rev. E **60**, 2721 (1999).
[15] G. E. Crooks, Phys. Rev. E **61**, 2361 (2000).
[16] T. Monnai, J. Phys. A **37**, L75 (2004).
[17] P. Gaspard, J. Chem. Phys. **120**, 8898 (2004).
[18] D. Andrieux and P. Gaspard, J. Chem. Phys. **121**, 6167 (2004).
[19] T. Monnai and S. Tasaki, e-print arXiv:cond-mat/0308337.
[20] T. Monnai, Phys. Rev. E **72**, 027102 (2005).
[21] T. Monnai, Nonlinear Phenom. Complex Syst. **10**, 103 (2007).
[22] D. Andrieux, P. Gaspard, T. Monnai, and S. Tasaki, New J. Phys. **11**, 043014 (2009).
[23] J. Kurchan, Report No. PMMH-ESPCI-34 (unpublished).
[24] W. Thirring, *Quantum Mathematical Physics; Atoms, Molecules and Large Systems* (Springer-Verlag, Wien, 2003).
[25] S. Tasaki and T. Matsui, in *Fundamental Aspects of Quantum Physics*, edited by L. Accardi and S. Tasaki (World Scientific, Singapore, 2003), p. 100.
[26] M. Esposito and S. Mukamel, Phys. Rev. E **73**, 046129 (2006).
[27] U. Harbola, M. Esposito, and S. Mukamel, Phys. Rev. B **76**, 085408 (2007).
[28] S. Deffner and E. Lutz, Phys. Rev. E **77**, 021128 (2008).
[29] I. Callens, W. De Roeck, T. Jacobs, C. Maes, and K. Netočný, Physica D **187**, 383 (2004); C. Maes and K. Netočný, J. Stat. Phys. **110**, 269 (2003).
[30] K. Saito and A. Dhar, Phys. Rev. Lett. **99**, 180601 (2007).
[31] K. Saito and Y. Utsumi, Phys. Rev. B **78**, 115429 (2008).
[32] A. E. Allahverdyan and T. M. Nieuwenhuizen, Phys. Rev. E **71**, 066102 (2005).
[33] M. Esposito, U. Harbola, and S. Mukamel, Rev. Mod. Phys. **81**, 1665 (2009).
[34] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, Phys. Rev. Lett. **89**, 050601 (2002).
[35] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys Rev A **37** 4419 (1988).
[36] F. Haake and R. Reibold, Phys. Rev. A **32**, 2462 (1985).
[37] C. W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991), Chap. 3, p. 4.
[38] C. W. Gardiner, IBM J. Res. Dev. **32**, 127 (1988).